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On Extremal Orthoposets Without Forbidden Substructures

Peter Brass[1]

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We study the structures of orthoposets forced by a combination of an extremum principle (maximum number of comparable pairs or of edges in the Hasse diagram) and an excluded substructure information. We show that in all interesting cases this reduces at least asymptotically to the corresponding graph problems, and we give the solution to some of these problems.

1. EXTREMUM PRINCIPLES FOR ORDER STRUCTURES

Extremum principles for the characterization of objects are an important type of mathematical question that arises in many physical applications, but up to now all these applications are either for analytic objects (variational principles) or geometric objects (isoperimetric-type theorems, e.g., soap bubbles, crystals, etc.). In the following we wish to give an example of how extremum principles can also be used for the characterization of order structures.

As in the classical cases, the problem is given by a ground set $\mathbb{O}$ of objects (e.g., orthoposets) on which a function $c: \mathbb{O} \to \mathfrak{R}$ (e.g., the number of comparable pairs) has to be maximized, subject to some restrictions. The restrictions define the set $\mathcal{A} \subset \mathbb{O}$ of admissible objects. Since our objects are ordered by a substructure relation, it is a natural assumption that the restrictions are given as forbidden substructures, i.e., as things that should not occur (e.g., 'no pentagons'). If $\mathcal{A}$ is closed under taking substructures, or c is monotone, then any restriction can be given that way. Also this presentation is useful, since any subset of the forbidden substructures generates an outer approximation of $\mathcal{A}$; comparing these approximations allows us to find out which restrictions are important for the problem.

[1] Institut für Mathematik; Universität Greifswald, D-17487 Greifswald, Germany.

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The main difficulty with this approach is to find a useful function c which is to be maximized. In fact, we have to restrict consideration in the following to finite structures (which are not physically interesting), since we do not know any reasonable function to maximize over infinite order structures. For finite structures we have the various counting functions, e.g., the numbers of comparable pairs, of edges of the Hasse-diagram, of 0–1 valued measures, etc. To make these have a maximum, we have to restrict the cardinality of the underlying poset. So we wish to maximize some counting function c over all orthoposets of $2n$ elements that do not contain some specified suborthoposets.

2. THE NUMBER OF COMPARABLE PAIRS IN AN ORTHOPOSET

In the following we will always study the set of orthoposets as object type $\mathcal{O}$. An orthoposet $(OP, \leq, \overline{})$ is the simplest ordered orthostructure: it consists of a poset $(OP, \leq)$ together with a self-mapping $\overline{} : OP \to OP$ which involutive ($\overline{\overline{x}} = x$), antitone ($x \leq y \Leftrightarrow \overline{y} \leq \overline{x}$), and an orthocomplement ($\neg\exists x, y\colon x \leq y \wedge \overline{x} \leq y$). Normally a minimum element 0 and a maximum element 1 are added (Flachsmeyer, 1988); we prefer this equivalent version since it allows nicer counting formulas (Brass, 1995). In the following we often use the number of elements as index, e.g., OP_{2n} denotes an orthoposet of $2n$ elements.

An orthoposet $(OP^*, \leq^*, \overline{}^*)$ is suborthoposet of $(OP, \leq, \overline{})$ if there is an injective map ω: $OP^* \to OP$ with $\omega(\overline{x}^*) = \overline{\omega(x)}$ and $x <^* y \Rightarrow \omega(x) < \omega(y)$.

The most obvious complexity measure for posets with a given number of elements is the number of comparable pairs $(x, y) \in OP \times OP$ with $x < y$; in the following we always count unordered comparable pairs, i.e., (x, y) and (y, x), only once. For this we have:

Theorem 1. The maximum number of pairs (x, y) with $x < y$ possible in an orthoposet with $2n$ elements is $n^2 - n$. All extremal orthoposets can be constructed as shown below.

The mapping $(x, y) \mapsto (x, \overline{y})$ is an involution of $OP \times OP$ that maps comparable pairs on incomparable pairs. If we remove the diagonal pairs (x, x) and the antidiagonal pairs $(x, \overline{x})$, which are interchanged by this mapping, the remaining pairs have to contain at least as many incomparable pairs as comparable pairs. Therefore an orthoposet OP with $2n$ elements contains at most $n^2 - n$ pairs (x, y) with $x < y$.

To describe the structure of all orthoposets that reach this bound, we have to give a definition. We call an orthoposet complete if it is of height two, and any lower element is comparable to any upper element unless they are complements of each other. So the complete orthoposet K_{2n}^{OP} of $2n$ elements consists of n lower elements $x_1, \ldots, x_n$ and n upper elements $\overline{x_1}, \ldots, \overline{x_n}$ with $x_i < \overline{x_j}$ iff $i \neq j$. The complete orthoposet K_{2n}^{OP} has $n^2 - n$ pairs $\{x_i, \overline{x_j}\} \subset K_{2n}^{\mathrm{OP}}$ with $x_i < \overline{x_j}$, so it is extremal.

To construct now from a given extremal orthoposet OP_{2m} with $2m$ elements a new extremal orthoposet OP_{2m+2k} with $2m + 2k$ elements, we select a minimum element $y \in OP_{2m}$, together with the corresponding maximum element $\overline{y}$, and join them to a K_{2k}^{OP} with the new elements $\{x_1, \ldots, x_k, \overline{x_i}, \ldots, \overline{x_k}\}$ by $x_i < y$, $\overline{x_i} > \overline{y}$ for all i, and $x_i < \overline{x_j}$ for $i \neq j$. This new orthoposet is again extremal, since each of the elements of OP_{2m} is comparable either to all x_i or to all $\overline{x_i}$, giving a total of $m^2 - m + k^2 - k + 2mk = (m + k)^2 - (m + k)$ comparable unordered pairs. Figure 1 shows the Hasse diagram of an orthoposet constructed in this way, the complementarity being given by a reflection along a horizontal line.

To show that each extremal set OP_{2n} can be obtained in this way, we note that in an extremal set there are the same number of comparable and incomparable pairs, so $(x, y) \mapsto (x, \overline{y})$ also maps incomparable pairs on comparable pairs. This implies in particular that any suborthoposet of height 2 must be a complete orthoposet. There are many such suborthoposets, for if A is an antichain, then $A \cup \overline{A}$ induces a suborthoposet of height 2. The lower elements of this suborthoposet again form an antichain with cardinality at least $|A|$.

Let now x and y be elements such that x is minimal, y is an upper neighbor of x, and all lower neighbors of y are also minimal elements. Suppose now z is another upper neighbor of x. Since z and y are incomparable, z must be comparable to $\overline{y}$; but $z < \overline{y}$ implies $x < z < \overline{y} < \overline{x}$, a violation of orthocomplementarity, so we have $z > \overline{y}$. Therefore all upper neighbors of x with the exception of y are maximal elements, and indeed upper neighbors of $\overline{y}$. So if X denotes the set of lower neighbors of y (with $x \in X$), then OP_{2n} is obtained from $OP_{2n} \setminus (X \cup \overline{X})$ by the construction described above, i.e., the orthoposet on $X \cup \overline{X}$ is complete, y is a minimal element in $OP_{2n} \setminus$

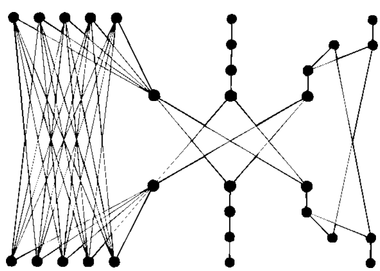

Fig. 1.

$(X \cup \overline{X})$, and we join each element of X as a lower neighbor to y and each element of $\overline{X}$ as an upper neighbor to $\overline{y}$.

3. EXCLUDED SUBSTRUCTURES AND SUBORTHOPOSETS OF HEIGHT TWO

Any structure interesting as an excluded substructure must be such that we can force it to occur in an orthoposet OP_{2n} as suborthoposet by the assumption of a lower bound on the number of comparable pairs in OP_{2n}. Therefore it must be a common substructure of all extremal orthoposets. Especially it must be of height 2, since the complete orthoposet is extremal and of height 2. Also, it has to occur in the two-chain orthoposet.

We can describe orthoposets OP_{2m} of height 2 by a graph $G(OP_{2m})$, with the complementary pairs $\{x, \overline{x}\}$ as vertices, and joining two vertices $\{x, \overline{x}\}$ and $\{y, \overline{y}\}$ by an edge iff there are comparable pairs in $\{x, \overline{x}, y, \overline{y}\}$. This defines a bijection between the graphs on m vertices and the orthoposets of height 2 with $2m$ elements, which maps the complete graph K_m on the complete orthoposet K_{2m}^{OP} [the orthoposet $OP(G)$ of a graph $G = (V, E)$ having elements $V \cup \overline{V}$, with $v < \overline{u}$ iff $\{u, v\} \in E$].

Now an orthoposet OP_{2m} of height 2 occurs in a two-chain orthoposet of cardinality $2n \geq 2m$ as suborthoposet iff its graph is bipartite, for any odd cycle in the graph would lead to a sequence of comparabilities connecting an element and its complement (which cannot occur if the orthoposet consists of two complementary chains). This is still not sufficient for OP_{2m} to occur in all extremal orthoposets, for the right orthoposet in Fig. 2 (with graph $K_{3,3}$) does not occur in the left one (which is extremal).

But this is an exception which does not occur if $2n$ is sufficiently big compared to the cardinality $2m$ of the excluded suborthoposet. We have:

Theorem 2. Let $n > 1/2(m - 1)^2$, and let OP_{2m} be an orthoposet with $2m$ elements which is of height 2 and has a bipartite graph. Then OP_{2m} occurs as suborthoposet in each extremal orthoposet OP_{2n} of $2n$ elements.

By Dilworth's theorem (Brass, 1995; Dilworth, 1950), each poset with at least $(m - 1)^2 + 1$ elements contains a chain of length m or an antichain

Fig. 2.

of cardinality m. These induce a two-chain suborthoposet or a complete suborthoposet of $2m$ elements each. But each orthoposet of height 2 with a bipartite graph occurs as a suborthoposet of the complete orthoposet as well as of the two-chain orthoposet of the same cardinality.

So for each height-2 poset OP_{2k} with bipartite graph G the question for the maximum number of comparable pairs in an orthoposet of $2n$ elements without OP_{2k} as suborthoposet is a nontrivial question. We can show that this is asymptotically the same as the corresponding question for graphs. The corresponding graph problem, the maximum number of edges in a graph with n vertices without G as subgraph, immediately gives a lower bound for the orthoposet problem. For, the bijection of orthoposets of height 2 and $2n$ elements and graphs with n vertices maps graph-edges $\{x, y\}$ on pairs $(x < y, \bar{y} < \bar{x})$ of comparable pairs, and maps subgraphs on suborthoposets. But orthoposets with bigger height cannot be much better. We have:

Theorem 3. Let G_k be a bipartite graph, $ex(n, G_k)$ the maximum edge-number of a graph with n vertices that does not contain G_k as a subgraph, and OP_{2k} the orthoposet of height 2 that corresponds to G_k. Then the maximum number of comparable pairs in an orthoposet with $2n$ elements that does not contain OP_{2k} as a suborthoposet is between $2 \cdot ex(n, G_k)$ and $\binom{k-1}{2} \cdot ex(2n, G_k)$.

Unfortunately, the problem of determining $ex(n, G_k)$ for bipartite G is notoriously difficult; it is known as the 'degenerate' case of the Turán problem (Simonovits, 1983). A global bound $ex(n, G_k) = O(n^{2-2/k})$ follows from the Kövari–Turán–Sós theorem $ex(n, K_{r,s}) = O(n^{2-1/s})$, since any bipartite graph G_k is subgraph of a complete bipartite graph $K_{r,s}$ with the same vertexset $(k = r + s)$. For more bounds on specific graphs see (Simonovits, 1983). An especially interesting case seems to be the case of an excluded 4-cycle C_4, since the orthoposets corresponding to the extremal graphs (Erdös–Renyi graphs; Füredi, 1996) are the lattices of the finite projective planes.

To prove Theorem 3, we note that the orthoposet OP_{2n} may not contain a chain of length k, otherwise it contains all orthoposets OP_{2k} that have bipartite graphs. So we may decompose the orthoposet OP_{2n} into $k - 1$ antichains $A_1, \ldots, A_{k-1}$ by repeatedly removing the sets of all maximal and of all minimal elements. Any comparable pair $x < y$ belongs now to a unique pair A_i, A_j of antichains with $x \in A_i$, $y \in A_j$. To count the comparable pairs in OP_{2n}, we have to bound the number of comparable pairs between any two of these antichains. For this we note that if the comparability graph of the height-2 poset defined on $A_i \cup A_j$ contains G_k as subgraph, then the orthoposet defined on $A_i \cup A_j \cup \overline{A_i} \cup \overline{A_j}$ contains the corresponding orthoposet OP_{2k} as suborthoposet. Therefore there are at most $ex(2n, G_k)$ comparable pairs between A_i and A_j. Taking the sum over all antichain pairs proves Theorem 3.

4. THE EDGE NUMBER OF THE HASSE DIAGRAM OF AN ORTHOPOSET

A complexity measure that is much easier to handle is the number of edges of the Hasse diagram. Each edge $\{x, y\}$ corresponds to a comparable pair $x < y$, so the number of edges is at most $n^2 - n$ in a $2n$-element orthoposet. This number is reached by the complete orthoposet K_{2n}^{OP}; and the complete orthoposet is the only extremal orthoposet, since all other OP_{2n} with maximum number of comparable pairs contain chains of length 3, and in a chain of length 3 there is a comparable pair that is not edge of the Hasse diagram. Since we have only one extremal structure, any suborthoposet of the complete orthoposet, i.e., any orthoposet of height 2 can be used as excluded substructure. By our bijection this directly mirrors the graph problem, since there is always an extremal orthoposet OP_{2n} (without OP_{2k} as suborthoposet) that is of height 2. For, we may find in each orthoposet OP_{2n} that is of height greater than 2 a suborthoposet OP_{2n}^{*} with the same elements and at least the same number of edges in the Hasse diagram, and smaller average height of the elements. We have just to pick a chain $x < y < z$ with x, z being neighbors of y, remove the comparability of x and y (losing one edge), and keep all others (especially those of x to all upper neighbors of y, which generates at least one new edge). So we have:

Theorem 4. Let G_k be a graph and OP_{2k} the orthoposet of height 2 that corresponds to G_k. The maximum edge number of the Hasse diagram of an orthoposet OP_{2n} that does not contain OP_{2k} as a suborthoposet is $2 \cdot ex(n, G_k)$.

For graphs G_k that are not bipartite these numbers are asymptotically known; by the Erdös–Stone theorem (Simonovits, 1983), they are essentially complete multipartite graphs with one class less than the chromatic number of the excluded subgraph. If the excluded subgraph is complete, the problem is solved by Turán's theorem (Simonovits, 1983): We may construct the orthoposet OP_{2n} with maximum edge number that does not contain a complete orthoposet K_{2k}^{OP} as suborthoposet by partitioning the lower elements $x_1, \ldots,$ x_n into $k - 1$ classes

$$\{x_1, \ldots, x_{\lfloor n/k-1 \rfloor}\}, \ldots, \{x_{\lfloor (k-2)n/k-1 \rfloor}, \ldots, x_n\}$$

of nearly equal cardinality and defining $x_i < \overline{x_j}$ iff x_i and x_j are in different classes. Figure 3 illustrates this for $k = 3$ and $n = 7$.

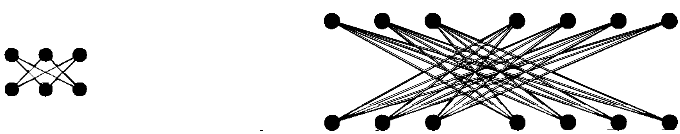

Fig. 3.

REFERENCES

Brass, P. (1995). Symmetric chain partitions of orthocomplemented posets, *International Journal of Theoretical Physics,* **34**, 1241–1245.

Dilworth, R. P. (1950). A decomposition theorem for partially ordered sets, *Annals of Mathematics,* **51**, 161–166.

Flachsmeyer, J. (1988). Orthoposets as substructures of Boolean algebras, *Topology and Measure V* (Binz/GDR 1987), Wiss. Beitr. Ernst-Moritz-Arndt-Universität Greifswald, pp. 89–94.

Füredi, Z. (1996). On the number of edges of quadrilateral-free graphs, *Journal of Combinatorial Theory Series B,* **68**, 1–6.

Simonovits, M. (1983). Extremal graph theory, in *Selected Topics in Graph Theory 2*, L. W. Beineke and R. J. Wilson, eds., Academic Press, pp. 161–200.